A Strongly Stable Implementation of Delayed State Derivative Feedback in Vibration Suppression

P. McGahan and T. Vyhlídal
CTU in Prague, Department of Control and Instrumentation Eng., Technická St. 4, 166 07 Praha 6, Czech Republic

April 24, 2007

Abstract

In this contribution, the problem of stability of a system with a state derivative feedback is studied. The emphasis is placed on the effects of small uncertain delays which originate as a result of latency phenomenon in the feedback. It is shown that the application of delayed state derivative feedback renders a system of the neutral type. This introduces difficulties with respect to system stability when small delay perturbations are considered and so, the notion of strong stability is utilised. A vibration suppression example is presented whereby the necessity of considering a strongly stable solution is shown. Furthermore, an optimisation procedure is used to search the eigenvalue space for potential strongly stable solutions. The resulting solution is optimised in the sense that the system eigenvalues are chosen to be as far from the stability boundary in the complex domain as possible and are subject to damping constraints.
Keywords: state derivative feedback; vibration suppression; strong stability

1 Introduction

Vibrations may be undesirable in dynamical systems for a host of reasons. They can affect product quality or functionality of products e.g. in the manufacturing of tools. They can affect system performance e.g. in delicate machines that may require vibration isolation. They can affect personal comfort e.g. vibrations in vehicle suspension systems. It is no surprise then that the need commonly arises for such vibrations to be suppressed in dynamical systems.

In vibration control problems, accelerometers are typically used for measuring the system motion. Accelerations are typically the sensed variables as opposed to displacements. When one considers feedback controller implementation, the question of acceleration feedback and indeed state derivative feedback naturally arises.

Much attention has been paid to this control problem in the literature. The application of acceleration feedback to vibration suppression problems has been discussed at length in [7] and [10]. A general pole placement technique for state derivative feedback was proposed in [11] for single-input systems and its application to vibration problems was emphasised. Following from this, the same authors proposed an LQR technique for computing state derivative feedback for multiple-input systems in [1]. The application of state derivative feedback to vibration problems has also been discussed from the perspective of robust control in [9].

Even when a dynamical system is modelled as ordinary differential equations, it is extremely important to consider latencies which arise from the application of a control action [5]. Such latencies can occur due to e.g. computational delays, AD-DA conversion, or communication delays. The role of latency phenomena on system stability is of crucial importance when one considers systems with state derivative feedback.

It is shown that the application of state derivative feedback renders a system of the neutral type. This induces complications with respect to system stability due to the fact that the system may be very sensitive even to infitessimal delay changes

[13]. For this reason, the notion of strong stability is utilised from [3] and [6]. This ensures robustness of stability w.r.t. delay perturbations.

The remainder of the paper is structured as follows: firstly a vibration suppression problem is presented as a motivating example. Then the pertinent issues of state derivative feedback and neutral equations are discussed. Finally, a case study is presented where the results of this study are analysed and compared to those in the literature.

2 Case Study

The vibration suppression example can be seen in figure 1. A linear state space formulation of the system, with states $x = [x_1 \; x_2 \; \dot{x}_1 \; \dot{x}_2]$ and assuming φ to be small, is given by

$$\dot{x} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1c_1 & -k_2c_2 & -b_1c_1 & b_2c_2 \\ -k_1c_2 & -k_2c_1 & -b_1c_2 & b_2c_1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c_1 & c_2 \\ c_2 & c_1 \end{pmatrix} u$$

where $c_1 = \frac{1}{m} + \frac{L^2}{I}$, $c_2 = \frac{1}{m} - \frac{L^2}{I}$, $x_3 = 0.5(x_1 + x_2$ and $\varphi = \frac{1}{2L}(x_1 - x_2)$. m and I are the mass and inertia of the mass respectively, k_1 and k_2 are the spring constants, b_1 and b_2 are the damping constants, $2L$ is the distance between the two supports, φ is the angle of inclination of the mass with the horizontal, x_3 is the displacement of the centre of the mass, x_1 and x_2 are the displacements of the sides of the mass and u_1 and u_2 are the control inputs. The objective of applying such control inputs is to interchange or dissipate kinetic and potential energy effectively such that system vibrations are reduced. One way of achieving this is by altering damping and stiffness characteristics of the system e.g. $b_{1,2}$ and $k_{1,2}$. Alternatively the inputs can simply displace the system thus counteracting the vibrations present and setting the system to rest.

The model parameters are given as $m = 10$kg, $I = 1kgm^2$, $L = 1m$, $k_1 = 500N/m$, $k_2 = 700N/m$, $b_1 = 10Ns/m$ and $b_2 = 20Ns/m$. It should be noted that the open loop system poles are $15.1384 \pm 31.1738j, 1.3616 \pm 10.7106j$.

The design objective is to achieve vibration suppression using a state derivative feedback controller. Firstly, it is necessary to consider the issue of state derivative feedback control design.

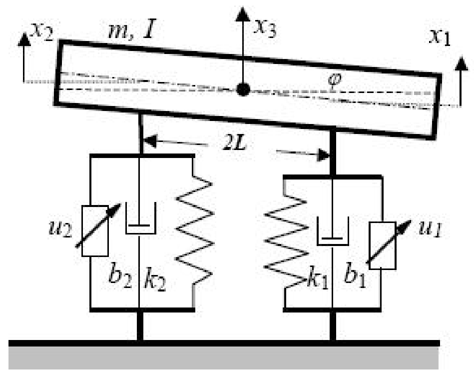

Figure 1: Vibration suppression example

3 State Derivative Feedback

Let us consider a system of the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \tag{1}$$

where $x \in \Re^n$ is the state and $u \in \Re^p$ are the inputs. It is assumed that $\mathbf{A}$ is of full rank, or in other words, the system is state controllable.

It is desired to design a stabilising controller of the form

$$u = -\mathbf{K}_{df}\dot{x} \tag{2}$$

where $\dot{x}(t)$ is the derivative of the states. The closed loop system then takes the form

$$\dot{x}(t) = (\mathbf{I} + \mathbf{B}\mathbf{K}_{df})^{-1}\mathbf{A}x(t) \tag{3}$$

The design problem is to compute a feedback gain matrix $\mathbf{K}$ such that the closed loop poles are located in the open left half complex plane. This is achieved through a system transformation to the Frobenius canonical form and by applying a technique similar to that applied in the derivation of the Ackermann formula for state feedback. For derivations of the Ackermann formula for state feedback and the equation for state derivative feedback the reader is referred to [8] and [11] respectively. The formula derived in [11] can be generalised, using work presented in [4] to compute state derivative feedback gains for MIMO systems.

It is important to note that a relationship can be derived between state feedback and state derivative feedback. Again the only requirement is that the open loop system is completely state controllable.

From [1], the following relationship can be derived

$$\begin{aligned}u(t) &= -\mathbf{K}_{sf}x = -\mathbf{K}_{df}\dot{x}\\ &= -\mathbf{K}_{df}(\mathbf{A}x + \mathbf{B}u)\\ &= -\mathbf{K}_{df}(\mathbf{A}x + \mathbf{B}(-\mathbf{K}_{sf}x))\\ &= -\mathbf{K}_{df}(\mathbf{A} - \mathbf{B}\mathbf{K}_{sf})x\end{aligned}$$

So the following relationship can be stated

$$\mathbf{K}_{df} = \mathbf{K}_{sf}(\mathbf{A} - \mathbf{B}\mathbf{K}_{sf})^{-1} \tag{4}$$

This is an interesting result considering the fact that algorithms for calculating state feedback gains are widely available in software packages such as MATLAB. This relationship is used to obtain the results presented in this work, unless otherwise stated.

4 Neutral Differential Equations

In real world systems, it is inevitable that small, uncertain time delays may occur in the feedback loop of an applied control algorithm. Recalling equation (1), and rewriting the equation considering such delays

$$\dot{x}(t) = \mathbf{A}x(t) + \sum_{j=1}^{N} \mathbf{B}_j u(t - \tau_j) \tag{5}$$

where $0 < \tau_1 < \tau_2 < \ldots < \tau_N$ are the delays arising in the feedback loop of the system. The summation over the input terms is necessary, due to the fact that each individual input may have a corresponding unique delay value. Now considering the state derivative feedback in (2), equation (5) can be written as

$$\dot{x}(t) + \sum_{j=1}^{N} \mathbf{B}_j K_j \dot{x}(t - \tau_j) = \mathbf{A}x(t) \tag{6}$$

The characteristic matrix of the system defined in equation (6) is given by

$$\Delta C(\lambda) = \left(\lambda \left(I + \sum_{j=1}^{N} \mathbf{B}_j K_j e^{-\lambda\tau_j}\right) - A\right) \tag{7}$$

Equation (6) is referred to as a neutral differential equation. A characteristic of neutral equations is that delays are present in the highest order derivative terms of the equation. Such a system description induces complications when one considers system stability. The concept of a strongly stable solution, introduced in [3], must therefore be considered.

4.1 Strong Stability

The associated difference equation of the system in (6) is given by

$$x(t) + \sum_{j=1}^{N} \mathbf{B}_j K_j x(t - \tau_j) \tag{8}$$

The characteristic matrix of the system defined in equation (8) is given by

$$\Delta D(\lambda) = \left(I + \sum_{j=1}^{N} \mathbf{B}_j K_j e^{-\lambda\tau_j}\right) \tag{9}$$

The necessary condition for system exponential stability can be stated from [2], that the the associated difference equation (8) must be exponentially stable in order for equation (6) to be exponentially stable. The following important relationship is given in [6]

$$r_e(T(1)) = r_\sigma(T_D(1)) \tag{10}$$

This means that the essential spectral radius of the equation (6) matches exactly the sprectral radius of the associated difference equation in equation (8). It is necessary to consider the smallest upper bound to the solution of the difference equation, given by

$$c_D(\vec{\tau}) = \sup \{\Re(\lambda) : \det(\Delta D(\lambda)) = 0\} \tag{11}$$

where $\vec{\tau}$ is a vector of the delays present in the system e.g. $\vec{\tau} = [\tau_1, \tau_2 \ldots \tau_n]$.

It is clearly stated in [3] that although the smallest upper bound (11) is continuous in the matrices $\mathbf{B}_j\mathbf{K}_j$, it is not continuous in the delays $\vec{\tau}$. A major consequence of this non-continuity is that arbitrarily small delay perturbations may destroy stability of the difference equation [6]. From [3], it can be stated that the solution of equation (8) is strongly exponentially stable if it remains exponentially stable when subjected to small variations in the delays. [2] and [3] propose the following condition.

The solution of the delay difference equation (8) is strongly exponentially stable iff $\gamma_0 < 1$, where

$$\gamma_0 := \max_{\vec{\theta} \in [0,\ 2\pi]^m} r_\sigma \left(\sum_{j=1}^{m} \mathbf{B}_j \mathbf{K}_j e^{i\theta_k} \right) \qquad (12)$$

Note:

- From [2] it can be stated that if $\gamma_0 > 1$ then equation (8) is exponentially unstable for rationally independent[1] delays.

- γ_0 does not depend on the value of the delays. This means that exponential stability locally in the delays is equivalent with exponential stability globally in the delays [5].

5 Case Study Results

In this section, results obtained in this work shall be analysed and compared to results obtained in the literature. The emphasis is placed on the destabilising effects of small uncertain delays and the resulting necessity to consider a strongly stable solution.

5.1 State Derivative Feedback

Recall the motivating example presented in section (2) and consider the stabilising state derivative feedback gain matrix from [1].

$$\mathbf{K}_{df} = \begin{pmatrix} 98.0081 & -6.5270 & 1.5875 & -1.5119 \\ -4.6622 & 35.6412 & -0.2978 & 1.9490 \end{pmatrix} \qquad (13)$$

The corresponding eigenvalues of the closed loop system are $(23.1893, 5.9365$ and $5.6331 \pm 10.1242j)$. The transient response from an initial state $x_0 = [-0.01\, 0.02\, -0.02\, 0.01]^T$ can be seen in figure (2). The system response shows desireable characteristics, namely a settling time of approximately $0.5s$, with moderate control effort.

[1]Delay terms e.g. $\vec{\tau}_1$ and $\vec{\tau}_2$ are deemed to rationally independent if their ratio is an irrational number. If the ratio is a rational number, the delay terms are considered to be rationally dependent.

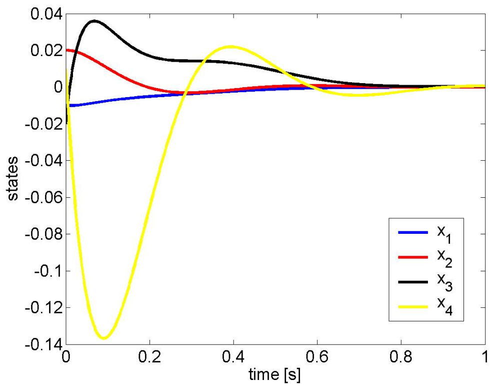

Figure 2: System response using state derivative feedback

5.2 Latent Effects and Strong Stablility

Now the effects of small uncertain delays in the feedback loop are analysed. Consider the system in equation (5), with $\vec{\tau} = [0.01\ 0.01\pi]$. Note that both delay terms are rationally independent. Equation (12) calls for the maximum of the function plotted in figure 3 to be found. Clearly in this instance, $\gamma_0 > 1$, so it can be stated that the system is exponentially unstable. This fact can be verified by analysing the spectrum of the system, using the quasipolynomial based rootfinding algorithm presented in [12], or alternatively through direct system simulation.

The algorithm presented in [12] consists of mapping the zero contours of the real and imaginary parts of the system characteristic equation and locating the intersection points of such curves. The intersection points are approximations to the system roots. The spectrum of the closed loop system, obtained by applying the algorithm from [12], can be seen in figure 5. It is evident from the figure that the system is unstable, due to the location of system roots in the open right half plane. The fact that the system is unstable is again seen in figure 4, where the response of the system displacements is shown to be tending towards $\pm\infty$.

It is important to recognise that the algorithms to compute state derivative feedback gains cited in this work [1, 11] do not consider the necessity of a strongly stable solution. From a practical application viewpoint, it is necessary to consider other

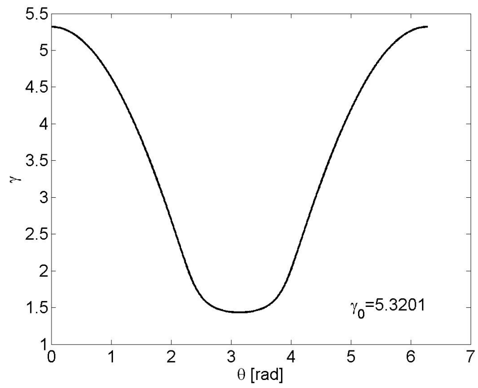

Figure 3: γ as a function of θ [rad]

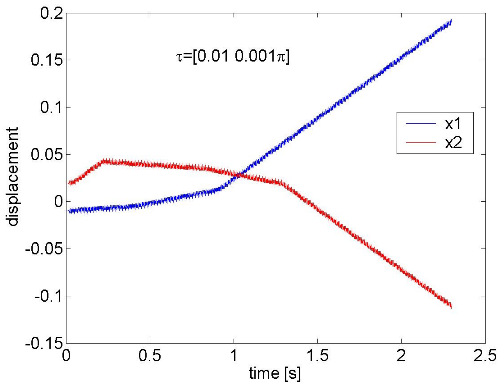

Figure 4: Response of system displacements, considering small time delays in feedback loop

methods of computing derivative feedback gains that directly consider strong stability.

5.3 Search Procedure

An optimisation procedure is used in this work to search the eigenvalue space for potential strongly stable stable solutions. The result is optimised in the sense that the solution with greatest robustness to small delay perturbations e.g. the solution with $min(\gamma_0)$ is chosen and is subject to damping constraints e.g. $(\zeta < 1)$.

- An upper and lower bound on $\Re(\lambda)$ are selected.
- A grid step d of the mesh is defined over the previously selected region.

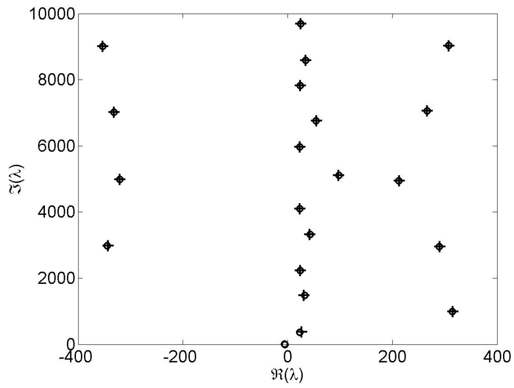

Figure 5: Spectrum of system, considering the effects of delays. The points correspond to the location of system poles from equation (6) whilst the '+' correspond to the solution of the associated difference equation in equation 8.

- Damping constraints corresponding to the line $\Im(\lambda) = -\Re(\lambda)$ are imposed on the search space.
- For an n^{th} order system, $n-1$ roots are held constant and the n^{th} root is incrementally moved along the mesh.
- At each incremental root change, the feedback gain matrix K_{df} and strong stability γ_0 are computed.
- The strongly stable solution with $min(\Re(\lambda))$ is chosen.

Returning to the motivating example, an upper bound of $\Re(\lambda) = -5$ and lower bound of $\Re(\lambda) = -35$ are selected. A grid step of $d = 3$ is selected and the damping contstraint is applied. The resulting search space is shown in figure 6.

In the interest of computational simplicity, the roots of the solution are chosen to be complex conjugates. The MATLAB algorithm *place* is used to compute the state feedback gain at each pole position and then the corresponding state derivative feedback gains are computed using the relationship in equation (4).

It is intersting to consider the variation of γ_0 w.r.t. changes in pole locations. In figure (7), γ_0 is plotted as a function of $\Re(\lambda_{3,4})$ and $|\Im(\lambda_{3,4})|$ while $\lambda_{1,2}$ are held constant. It is interesting to note that

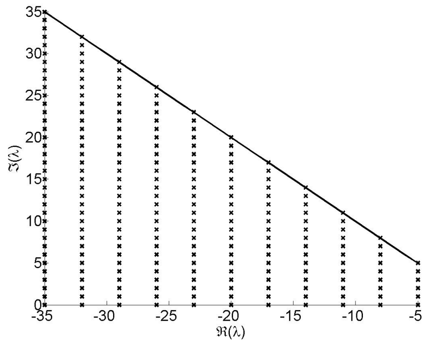

Figure 6: Search space

only an area of high frequency poles close to the origin give rise to an exponentially unstable solution. This information would suggest that the solution can be arbitrarily chosen outside of this region. In the interest of solution robustness however, the solution with $min(\gamma_0)$ is chosen instead. On comple-

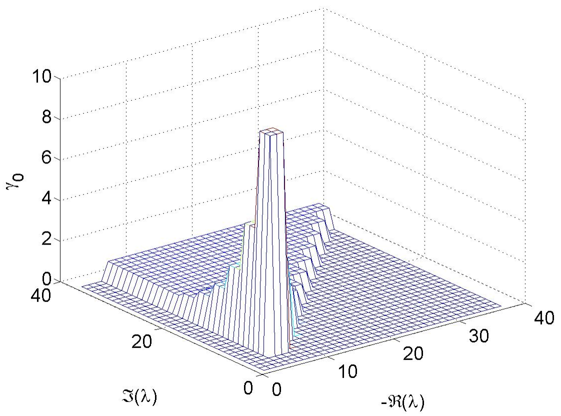

Figure 7: Variation of γ_0 w.r.t. one pair $\lambda_{3,4}$ of complex conjugate system roots. $\lambda_{1,2} = -23 \pm 17i$.

tion of the search procedure, the following solution was obtained.

- Closed loop system roots
  $\lambda_{1,\dots,4} = [-20 \pm 11i, \ -23 \pm 17i]$
- Feedback gain matrix
  $K_{df} = \begin{pmatrix} 28.3877 & -0.0038 & -1.7903 & -2.2501 \\ -0.0053 & 19.3643 & -2.2502 & -1.8943 \end{pmatrix}$
- Strongly stable solution where $\gamma_0 = 0.8191$

The transient response of the system from the same initial condition $x_0 = [-0.01\ 0.02\ -0.02\ 0.01]^T$ and again taking $\vec{\tau} = [0.01\ 0.001\pi]$, is given in figure 8. The closed loop system shows good response characteristics, even in the presence of small time delays. A settling time of approximately $0.65s$ is observed.

As a basis for comparison, the spectrum of the closed loop system, under the influence of the small time delays, with a feedback gain matrix obtained via the search procedure is presented in figure 9.

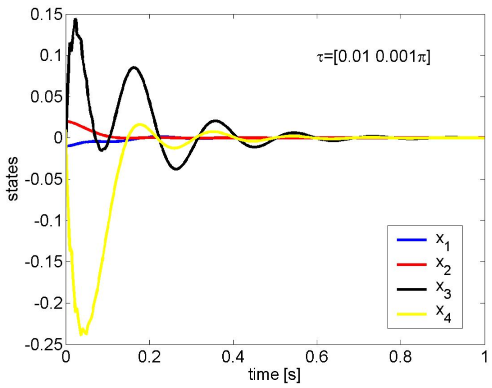

Figure 8: Transient response of system with state derivative feedback gain matrix computed using search procedure

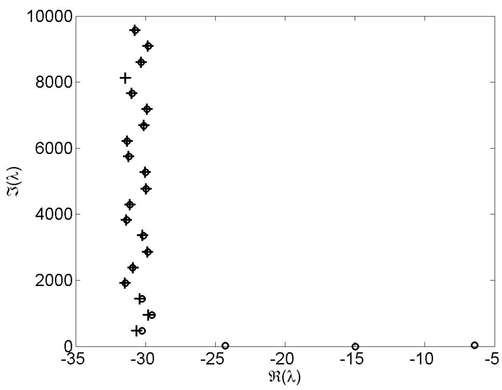

Figure 9: Spectrum of system, after search procedure, considering the effects of delays.

6 Conclusions

The effects of small uncertain delays in the feedback loop of systems controlled by a state derivative feedback was studied. It was shown that such inevitable delays can adversely affect system stability if not accounted for by using the concept of strong stability. A vibration suppression example was presented whereby the importance of considering a strongly stable solution was highlighted.

References

[1] T.H.S Abdelaziz and M. Valášek. State derivative feedback by lqr for linear time-invariant systems. In *IFAC 16th Triennial World Congress*, 2005.

[2] J.K. Hale and S.M. Verduyn Lunel. *Introduction to functional differential equations*. Applied Mathematical Sciences New York: Springer, 1993.

[3] J.K. Hale and S.M. Verduyn Lunel. Strong stabilization of neutral functional differential equations. *IMA Journal of Mathematical Control and Information*, (19):5–23, 2002.

[4] N. Olgac M. Valá˘ sek. Efficient eigenvalue assignments for general linear mimo systems. *Automatica*, 31(11):1605–1617, 1995.

[5] W. Michiels. *Stability and Stabilization of Time-Delay Systems*. PhD thesis, Department of Computer Science, K.U. Leuven, 2002.

[6] W. Michiels and T. Vyhlídal. An eigenvalue based approach for the stablization of linear time-delay systems of neutral type. *Automatica*, (4):991–998, 2005.

[7] M. Hosek N. Olgac, H. Elmali and M. Renzulli. Active vibration control of distributed systems using delayed resonator with acceleration feedback. *Transactions of ASME Journal of Dynamic Systems, Measurement and Control*, (119):380, 1997.

[8] K. Ogata. *Modern Control Engineering*. Pearson Prentice Hall, fourth edition, 2002.

[9] E. Reithmeier and G. Leitmann. Robust vibration control of dynamical systems based on the derivative of the state. *Archive of Applied Mechanics*, (72):856864, 2003.

[10] P Quast MK Sain DC Kaspari SJ Dyke, BF Spencer and TT Soong. Acceleration feedback control of mdof structures. *Journal of Engineering Mechanics*, (122):907–917, 1996.

[11] M. Valá˘ sek T.H.S. Abdelaziz. A direct algorithm for pole placement by state derivative feedback for single-input linear systems. *Acta Polytechnica*, 43(6):52–60, 2003.

[12] T. Vyhlídal. *Analysis and Synthesis of Time Delay System Spectrum*. PhD thesis, Control and Systems Engineering, CTU in Prague, 2003.

[13] D. Roose W. Michiels, K. Engelborghs and D. Dochain. Sensitivity to infinitesimal delays in neutral equations. *SIAM Journal of Control and Optimization*, 4(40):1143–1158, 2002.