Modification of Symmetric Optimum Method

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Abstract: This contribution deals with a modification of the symmetric optimum method. This method is designated for the synthesis of linear one-dimensional regulatory circuits whose structure can be divided into a controller and a controlled system. It enables the design of continuous controllers only. New equations for specific combinations of controllers and controlled systems been derived, which provide an approach for designing both the continuous and discrete controllers. Derivations have been carried out based on delta models principle.

Keywords: synthesis, controllers, delta models

1 A synthesis of linear one-dimensional regulatory circuits

First we have to specify what we understand with the terms regulatory circuit and synthesis of regulatory circuits. In this case we will be dealing with the synthesis of linear one-dimensional regulatory circuits whose structure can be divided into a controller and a controlled system [Balátě, 2003]. The synthesis process deals with designing adjustable controller parameters.

![Figure 1 – Regulatory circuit](image)

2 The symmetric optimum method

The symmetric optimum method is especially suitable for a case when the transfer function of an open regulatory circuit has a third degree multinominal polynomial in the denominator and the number of integrators \(q = 2\). First, we will describe the derivation process of an equation for the calculation of adjustable controller parameters by L-transform.

We will have a controlled system which will be expressed by the transfer function:

\[
G_s(s) = \frac{k_i}{s(T_i s + 1)},
\]

and we will choose a PI controller for this controlled system:

\[
G_c(s) = k_p \left(1 + \frac{1}{T_i s}\right).
\]

Then the transfer function of this opened regulatory circuit is
The symmetric optimum method is based on the general equation
\[ \sum_{i,j} (a_{ij} - a_{ji}) = 0, \]  
and the transfer function of the closed regulatory circuit is
\[ G_c(s) = \frac{k_1 k_p T_I s + k_1 k_p}{s^2 (T_I T_I s + T_I)} = \frac{a_1 s + a_0}{s^2 (a_1 s + a_2)}, \]  
where
\[ A_i = a_i^2 + 2 \sum_{j=1}^{m} (-1)^j a_{i-j} a_{i+j}. \]

It means, the determination of the adjustable controller parameters will be based on the system of equations
\[ A_i = 0, \quad i = 1, 2, ..., m, \]  
where \( m \) is the number of chosen adjustable controller parameters. In our case \( m = 2 \) and so we will solve these equations
\[ A_1 = (k_1 k_p T_I)^2 - 2 k_1 k_p T_I = a_1^2 - 2 a_1 a_2 = 0, \]
\[ A_2 = T_I^2 - 2 k_1 k_p T_I^2 = a_2^2 - 2 a_1 a_2 = 0, \]
where \( a_0, a_1, a_2 \) are coefficients of characteristic multinomal.

When we solve these equations we obtain two equations for calculating the adjustable controller parameters
\[ k_p^* = \frac{1}{2k_1 T_I}, \]
\[ T_I^* = 4 T_I. \]

After substitution into (3) and (4) we obtain the transfer function of the open regulatory circuit in standard form
\[ G_o(s) = \frac{4 T_I s + 1}{8 T_I^2 s^2 (T_I s + 1)} \]
and the transfer function of the closed regulatory circuit in standard form for symmetric optimum method
\[ G_c(s) = \frac{4 T_I s + 1}{8 T_I^3 s^3 + 8 T_I^2 s^2 + 4 T_I s + 1} = \frac{4 T_I s + 1}{(2 T_I s + 1)(4 T_I^2 s^2 + 2 T_I s + 1)}. \]

The transient characteristic of closed regulatory circuit is shown in figure 2.
3 Derivation of equations for calculation adjustable parameters of controller based on $\delta$ models

We will have the same transfer functions of a controlled system and controller as in previous case.

We will carry out discretization of equation (1). Discretization is based on this equation

$$G_s(s) = \frac{\gamma}{1 + T\gamma} D \left\{ L^{-1} \left[ \frac{1}{s} G_w(s) \right] \right\},$$  

(12)

Then we will obtain discrete $D$ - transfer function of the controlled system

$$G_s(y) = k_i \left( \frac{T - bT_1}{\gamma(T\gamma + b)} \right), \quad b = 1 - e^{\frac{T}{T_1}}.$$  

(13)

Now we need to obtain $D$ – the transfer function of the controller. The relation between the input and output of the PS controller using backward rectangle summation is

$$u(kT) = k_p \left\{ e(kT) + \frac{T}{T_i} \sum_{i=0}^{k} e(iT) \right\}.$$  

(14)

For transfer we will utilize this property of $D$ - transformation

$$D \left\{ T \sum_{i=0}^{k} x(iT) \right\} = \frac{1 + T\gamma}{\gamma} X(\gamma).$$  

(15)

After transfer we will obtain

$$U(\gamma) = k_p \left( \frac{1 + \gamma T}{T_i\gamma} \right) E(\gamma).$$  

(16)

We express $D$ – transfer function of the PS controller

$$G_s(\gamma) = \frac{U(\gamma)}{E(\gamma)} = k_p \left( \frac{1 + \gamma T}{T_i\gamma} \right).$$  

(17)

In order to determine the characteristic multinominal and its coefficients we need to express either the transfer function of the closed regulatory circuit or the transfer function of the open regulatory circuit. We will use second way

$$G_o(\gamma) = G_s(\gamma) G_s(\gamma) = \frac{M_o}{N_o} = k_i k_p \frac{\left( T - bT_1 \right)\gamma + b \left( T\gamma + T\gamma + 1 \right)}{T_i\gamma^2 (T\gamma + b)}.$$  

(18)

Now we can express the characteristic multinominal which is generally defined

$$N(\gamma) = M_o + N_o = a_n \gamma^n + a_{n-1} \gamma^{n-1} + \ldots + a_i \gamma + a_0.$$  

(19)

In our case we obtain

$$N(\gamma) = T T_i \gamma^3 + \left[ b T_i + k_i k_p \left( T T_i - b T_i T_i + T^2 - b T_i T \right) \right] \gamma^2 + k_i k_p \left( T - b T_i + b T_i + b T \right) \gamma + b k_i k_p.$$  

(20)

We express coefficients $a_0, a_1, a_2$ and $a_3$ from this equation

$$a_0 = b k_i k_p,$$  

(21)

$$a_1 = k_i k_p \left( T - b T_i + b T_i + b T \right),$$  

(22)

$$a_2 = b T_i + k_i k_p \left( T T_i - b T_i T_i + T^2 - b T_i T \right),$$  

(23)

$$a_3 = T T_i.$$  

(24)

We can substitute coefficients $a_0, a_1$ and $a_2$ into the equation for calculating $A_i$. We will express the general equation for calculating the gain $k_p^*$ of the controller from this equation.

$$k_p^* = \frac{2b^2 T_i^*}{k_i \left( b^2 T_i^{*2} + 2b^2 TT_i^* + b^2 T_i^{*2} - 2b T_i T_i^* + b^2 T_i^* + T_i^* \right)^2}.$$  

(25)
We substitute this equation into (22), (23) and (24) and coefficients $a_1, a_2$ and $a_3$ substitute into equation for calculation $A_2$. We will express general equation for calculation $T_i^*$ from this equation

$$T_i^* = \frac{1}{b} \left[ \frac{O}{3} - \frac{3}{9} \left( \frac{8}{9} T^2 - \frac{8}{3} bT_i \right) + bT_i - bT + \frac{1}{3} T \right]$$

$$O = \sqrt[3]{-72bT^2T_i + 108b^2T_i^2 + 28T^3 + P}.$$  (26)

$$P = 12\sqrt[3]{-204b^3T_i^3 + 174b^2T^4T_i^2 - 60bT^5T_i + 9T^6 + 81b^4T^2T_i^4}.$$  

From equations (25) and (26) it is evident that the derivation of these equations is quite complicated and doing that manually takes too much time. The symbolic mathematic toolbox in MATLAB was used for this derivation.

In order to simplify these equations we used the approximation

$$b = 1 - e^{-\frac{T}{T_i}} \approx \frac{2T}{(2T_i + T)}. \quad (27)$$

We obtained the equation for calculating $k_p^*$ in this form after approximation (27) into (25)

$$k_p^* = \frac{8T_i}{k_1 (4T_i^2 + 8T_iT + 5T^2)}.$$  \quad (28)

and equation for calculation $T_i^*$ in this form after approximation (27) into (26)

$$T_i^* = \frac{1}{6\sqrt[3]{Q}} \left[ \sqrt[3]{2\sqrt[3]{Q^2} + \sqrt[3]{(2T_i + T) + 16\sqrt[3]{2RT_i - 4\sqrt[3]{2RT} + 8T_i\sqrt[3]{Q} - 5\sqrt[3]{QT}}} \right]$$

$$Q = 64T_i^2 - 8T_iT + 7T^2 + 3\sqrt{3T}\sqrt{64T_i^2 - 16T_iT + 3T^2}.$$  \quad (29)

$$R = \frac{1}{3}(2T_i + T)^2.$$  

Now we need to simplify equations (28) and (29). We can rewrite equation (29) into this form

$$T_i^* = \frac{8192T_iR}{6Q} - \frac{128TR}{6Q} + \frac{4\sqrt[3]{2T_i + T}}{6} + \frac{4}{3} T_i - \frac{5}{6} T.$$  \quad (30)

After analysis of the first three members of equation (30) we can express this equation in following form

$$T_i^* = \frac{4}{3} T_i + \frac{5}{24} T - \frac{2}{6} T + \frac{4}{3} T_i + \frac{11}{24} T + \frac{4}{3} T_i - \frac{5}{6} T.$$ \quad (31)

We will obtain the simplified equation for calculation $T_i^*$ after last modification

$$T_i^* = 4T_i - \frac{T}{2}. \quad (32)$$

Now we simplify (28). First we neglect $5T^2$ and then we substitute (32) and we will obtain simplified equation for calculation $k_p^*$

$$k_p^* = \frac{4}{k_1 (8T_i + 3T)}.$$ \quad (33)

If we consider the limit case $T \rightarrow 0$ then we obtain system of equation as same as we obtained in derivation when we used L - transformation
\[ k_p^* = \frac{1}{2k_1 T_1}, \quad (34) \]
\[ T_i^* = 4T_1. \quad (35) \]

4 Example

We will have the mathematic model of controlled system in this form

\[ G_s(s) = \frac{1}{s(s + 1)}. \quad (36) \]

First we will choose a PI controller. The sampling period will be \( T = 0 \) in this case. Comparison of calculated adjustable parameters of controller is stated in Table 1.

<table>
<thead>
<tr>
<th>( T_i^* )</th>
<th>( k_p^* )</th>
</tr>
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<tbody>
<tr>
<td>equation (29)</td>
<td>4</td>
</tr>
<tr>
<td>equation (32)</td>
<td>4</td>
</tr>
<tr>
<td>equation (28)</td>
<td>0.5</td>
</tr>
<tr>
<td>equation (33)</td>
<td>0.5</td>
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Now we choose a PS controller and the sampling period will be \( T = 0.3 \). Comparison of calculated adjustable parameters of controller is stated in Table 2.

<table>
<thead>
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<th>( T_i^* )</th>
<th>( k_p^* )</th>
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<tbody>
<tr>
<td>equation (29)</td>
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<tr>
<td>equation (32)</td>
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<tr>
<td>equation (28)</td>
<td>0.4475</td>
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<tr>
<td>equation (33)</td>
<td>0.4495</td>
</tr>
</tbody>
</table>

Subsequently we performed numerical simulation in the MATLAB environment. We see the transient characteristic in Figure 3. We can see presumptive overshoot 43.3% in case when we used PI controller and sampling period was \( T = 0 \). In the second case when we the used PS controller the overshoot is bigger. This is logical because of the sampling period.
5 Conclusion

In this contribution we dealt with the modification of the symmetric optimum method. We explained what we understand with the term regulatory circuit and synthesis of regulatory circuits. Subsequently we explained when the symmetric optimum method is suitable. Then we showed the derivation of the equations for calculating the adjustable controller parameters using L – transformation and subsequently by the means of delta models. The next goal is to derive the equations for calculating the adjustable controller parameters for other combinations of controlled system and controller.

6 References