On continualisation in new approach to system theory

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Abstract: The general system theory is a natural basis of cybernetics as well as informatics. However, none of the contemporary system theories has been accepted by professional community up to now. A fundamentally new approach to system theory has been recently presented. The system definitions are based on quite new system paradigms stemming from attentive observations and resulting in an axiomatic system theory with correctly and uniquely defined notions. System behavior is supposed to be generally stochastic containing deterministic behavior as its special case. Quite newly and independently on behavior are treated causal dependencies among system variables, which leads into unique definitions of system structure notions. A necessity to study the discrete-time systems as original ones follows from these new axioms and from the causality law while the continuous-time systems are regarded as limits of suitable sequences of discrete-time systems. This limit process is called the continualisation procedure and must be treated very well because of the system behavior possible changes. The new approach provides surprising clear and easy solution of various cybernetic problems even such ones considered so far as insoluble.

Keywords: Kalman filter, optimal estimation, stochastic causal system, system theory, singular uncertainty

1 Introduction

Although the general system theory, intensively studied since sixties (Zadeh, Desoer, 1963), (Kalman, Falb, Arbib, 1969), (Klir, 1972), (Mesarovic, Takahara, 1989) is a natural basis of cybernetics, no theory has been generally accepted by professional community so far. The fact that there are strong contradictions with observations in known system theories is considered to be a main reason for it. This is why some authors define a system rather by examples than by an exact definition and why the others have abandoned system approach completely and use ad hoc models with no claim to cope with general system properties at all. Cybernetics thus loses its firm foundation and become rather a set of service instructions than a serious theory.

2 New approach to system theory

A reason of unsatisfactory results of contemporary system theories can be seen namely in the fact that they do not discern directional and non-directional causation among system variables (Žampa, Arnošt, 2000), (Willems, 1991).

The system can be meaningfully defined as an assemble of all interconnected subsystems together with their sources. The new system theory is then based on following paradigms.
A fundamental notion of the system theory is the system, which has to be viewed as a certain entire whole enabling to find not only behavioral equations but also direction of all causal relations. It is, therefore, supposed that the real system is such a part of the Universe, which can be completely described within the part that is not affected by its surroundings. This paradigm leads to the system that has no external inputs.

With respect to the previous paradigm, system processes cannot be obviously stimulated by system surroundings. Therefore, it has to be supposed that the system processes are generated by an internal system mechanism, which is supposed to be governed by a newly viewed principle and law of causality.

According to our knowledge of physics it is supposed that the system behavior is generally stochastic and may be approximated by deterministic behavior in justified cases. Remind, however, that it is spoken about a system approximation, not about a subsystem approximation as the latter generally depends on surroundings of the subsystem. Systems different from the stochastic ones are thus supposed not to be feasible in our physical world. As our knowledge of the Universe contains only a finite number of independent observations it has to be assumed that the domain of any system variable contains at most a finite number of elements. An extension to infinite sets cannot be based on observations but has to be postulated by appropriate hypotheses.

The system is generally supposed to be composed of parts mutually interconnected only by directional (one-way) interconnections, which can be disconnected and/or reconnected by an external intervention. These parts are called subsystems. Subsystems defined in this way are defined together with their interconnections. Their properties as well as properties of their admissible interconnections are then definable quite precisely as they are given as uniquely definable parts of the exactly defined original system.

These paradigms lead to the axioms defining directly the discrete-time systems only. The continuous-time system is then defined as a limit of a suitable sequence of previously defined discrete-time systems (Žampa, 1997). The progressive process of continualisation is presented in the following text as well as some advantages of this approach shown on an application.

3 Continualisation of stochastic causal systems

From the above stated new system paradigms follows that the continuous-time systems can be studied only as limits of suitable sequences of previously defined discrete-time systems. This is caused by the fact that we are able to obtain only a finite number of independent observations while studying the system properties in the real world. If a continuous-time system is given as a part of a cybernetic task, it is necessary to find such a sequence of discrete-time systems, which will converge to the given continuous-time system. This extension to infinite sets cannot be based on observations but has to be postulated by appropriate limit process called the continualisation procedure.

Since the terms of causality law, causal probability, causal function are losing their meanings easily defined in the discrete-time domain, the full attention has to be given to the process of system continualisation. In the following text, the principles of continualisation procedure will be shown and the system behavior will be studied in terms of the new approach to system theory.

Let a time-invariant linear stochastic system $S_0$

$$S_0 : \quad x_0(t + h_0) = A_0 x_0(t) + \Gamma_0 \cdot \xi(t + h_0)$$

(1)
is defined in the time set $T_0$

$$T_0 = \{0, h_0, 2h_0, 3h_0, \cdots \},$$

where $\{\xi_0(t), t \in T_0\}$ is a sequence of mutually independent stochastic variables $\xi_0(t)$ so that

$$\xi_0(t) \approx N(0, I) \text{, } t \in T_0$$

and $\xi_0(t)$ is also independent on generally stochastic initial condition

$$x_0(t) \approx N(m_0(0), Q_0(0)),$$

where $m_0(0)$ is a finite mean value and $Q_0(0)$ is a finite covariance matrix of $x_0(0)$. Our aim is to extend the set $T_0$ to the set $T = (0, +\infty)$, so that the properties of system $S$ are stochastically equal to the properties of system $S_0$ defined in $T_0 \subset T$.

The continuous-time system $S$ is defined as the limit of discrete-time system sequence $S_k$ given by

$$S_k: \quad x_k(t + h_k) = A_k x_k(t) + \Gamma_k \cdot \xi_k(t + h_k)$$

in the time set $T_k = \{0, h_k, 2h_k, 3h_k, \cdots \}$, where $h_k = \frac{h_0}{2^k}$, $h_0 > 0$.

The sequence $\{\xi_k(t), t \in T_k\}$ consists of stochastic variables $\xi_k(t)$ so that

$$\xi_k(t) \approx N(0, I) \text{, } t \in T_k$$

and $\xi_k(t)$ are mutually independent and independent on initial conditions

$$x_k(t) \approx N(m_k(0), Q_k(0)),$$

where $m_k(0)$ is a finite mean value and $Q_k(0)$ is a finite covariance matrix of $x_k(0)$.

The stochastically equal properties of $x_k(t)$ from $S_k$ and $x_0(t)$ from $S_0$ are demanded in $T_0 \subset T_k$ for finite $k = 1, 2, 3, \ldots$. Then the limit trajectory properties of $x(t)$ are expected to be stochastically equal to the trajectory properties of $x_0(t)$ provided the limit trajectory exists.

The equal stochastic properties of systems $S_{k+1}$ and the system given by Eq. 1 must be necessarily ensured to determine the sequence of discrete-time systems for finite $k = 1, 2, 3, \ldots$ in the time set $T_k$

$$S_{k+1}: \quad x_{k+1}(t + h_{k+1}) = A_{k+1} x_{k+1}(t) + \Gamma_{k+1} \cdot \xi_{k+1}(t + h_{k+1}).$$

Let us substitute the time-point $t \in T_{k+1}$ by $t + h_{k+1} \in T_{k+1}$ in Eq. (3) to find the stochastic equivalence conditions. Hence, we obtain

$$x_{k+1}(t + 2h_{k+1}) = A_{k+1} x_{k+1}(t + h_{k+1}) + \Gamma_{k+1} \cdot \xi_{k+1}(t + 2h_{k+1}).$$

From Eq. (3) and Eq. (4) follows

$$x_{k+1}(t + 2h_{k+1}) = A^2_{k+1} x_{k+1}(t) + A_{k+1} \cdot \Gamma_{k+1} \cdot \xi_{k+1}(t + 2h_{k+1}) + \Gamma_{k+1} \cdot \xi_{k+1}(t + 2h_{k+1}).$$

As $2h_{k+1} = h_k$ holds, hence

$$x_{k+1}(t + h_k) = A^2_{k+1} x_{k+1}(t) + A_{k+1} \cdot \Gamma_{k+1} \cdot \xi_{k+1}(t + h_k) + \Gamma_{k+1} \cdot \xi_{k+1}(t + h_k).$$

If the Eq. (6) is supposed to be equal to the Eq. (1), then the initial conditions must be equal for every finite $k = 1, 2, 3, \ldots$

$$m_{k+1}(0) = m_k(0),$$
$$Q_{k+1}(0) = Q_k(0).$$

The mean and covariance matrix of given Eq. (6) and (1) must be also equal. Hence

$$A^2_{k+1} = A_k.$$
\[
\Gamma_k \Gamma_k^T = A_{k+1} \Gamma_{k+1} \Gamma_{k+1}^T + A_{k+1}^T \Gamma_{k+1}. 
\] (10)

The Eq. (7), (8), (9) and (10) are differential equations with initial conditions \( A_0, \Gamma_0, m_0(0), Q_0(0) \) given by the original system \( S_0 \). These equations determine parameters of the discrete-time sequence of systems \( S_n, n = 1, 2, 3, \ldots \). The solution of Eq. (7) and (8) for every finite \( n \) gives the initial conditions as
\[
m_n(0) = m_0(0),
\] (11)
\[
Q_n(0) = Q_0(0).
\] (12)

Furthermore, from Eq. (9) it follows that
\[
A_n^{\varphi} = A_0,
\] (13)
and thus
\[
A_n = A_0^{\varphi},
\] (14)
for every \( n = 1, 2, 3, \ldots \).

The matrix \( \Gamma_n \) is given by solution of Eq. (10).

In the stated above, we have found the matrices \( A_k, \Gamma_k \) of the system sequence
\[
x_k(t + h_k) = A_k x_k(t) + \Gamma_k \cdot \xi_k(t + h_k),
\] (15)
with initial conditions \( x_k(t) = N \{ m_k(0), Q_k(0) \} \) independent on the index \( k \) for given \( A_0, \Gamma_0 \).

Hence, we can rewrite the Eq. (15) as
\[
x_k(t + h_k) = A_0^{\varphi} x_k(t) + \Gamma_k \cdot \xi_k(t + h_k)
\] and after subtracting \( x_k(t) \) as follows
\[
x_k(t + h_k) - x_k(t) = (A_k - I)x_k(t) + \Gamma_k \cdot \xi_k(t + h_k).
\] (16)

Consider now \( \Delta x_k(t) = x_k(t + h_k) - x_k(t) \) and \( \Delta t_k = h_k \).

Finally, the Eq. (16) can be rewritten as
\[
\Delta x_k(t) = \frac{1}{\Delta t_k}(A_k - I)x_k(t) \cdot \Delta t_k + \frac{1}{\sqrt{\Delta t_k}} \Gamma_k \sqrt{\Delta t_k} \cdot \xi_k(t + \Delta t_k).
\] (17)

This stochastic equation can be limited in terms of the limit in the mean (Srinivasan, Mehata, 1988) for \( k \to +\infty \). Hence, the following equation was derived.
\[
dx(t) = F \cdot x(t)dt + G \cdot dw(t),
\] (18)
where
\[
dx(t) = \lim_{k \to +\infty} \Delta x_k(t),
\] (19)
\[
dt = \lim_{k \to +\infty} \Delta t_k,
\] (20)
\[
dw(t) = \lim_{k \to +\infty} \Delta w_k(t);
\] \( \Delta w_k(t) = \sqrt{\Delta t_k} \cdot \xi_k(t + \Delta t_k) \), (21)
\[
F = \lim_{k \to +\infty} \frac{1}{\Delta t_k}(A_k - I),
\] (22)
\[
G = \lim_{k \to +\infty} \frac{1}{\sqrt{\Delta t_k}} \Gamma_k.
\] (23)

Eq. (21) follows from (Žampa, 2000), where \( w(t) \) is regarded as the Wiener-Lèvy stochastic process.

On the other hand, we can easily prove that the Eq. (2) can be obtained by the solution of the stochastic differential equation (18). Hence, the matrix \( A_k \) is given by
or with approximation of the Taylor series expansion to second order as

\[ A_k = I + F \cdot h_k + \frac{1}{2} F^2 \cdot h_k^2 + \cdots, \]  

(24)

the matrix \( \Gamma_k \) is then postulated as

\[ \Gamma_k = \sqrt{h_k} \cdot G. \]  

(26)

Eq. (24) or (25) and (26) can be used to get the sequence of discrete-time systems Eq. (2) from a given continuous-time system Eq. (18) for every finite \( k = 1, 2, 3, \ldots \).

### 4 Application

The advantages of the new approach to system theory will be shown on an estimation example, solution of which still makes some problems in contemporary system theories. Consider a stochastic system

\[ \begin{align*}
&dx(t) = F \cdot x(t) dt + G \cdot dw(t) \\
&dy(t) = H \cdot x(t) dt + Q \cdot dw(t),
\end{align*} \]

(27)

where \( x(t) \) is a vector of non-measurable state variables, \( F \) is a matrix of the appropriate order, \( y(t) \) is a vector of measurable state variables, \( G \) and \( Q \) are rectangular matrices of \( w \) regarded as Wiener-Lévy stochastic process with \( E\{dw\} = 0 \) and \( E\{dw dw^T\} = I dt, QQ^T \) is singular.

The previous equation can be rewritten, as a special case, into components of the vector \( x_k(t) \) as follows

\[ \begin{align*}
&dx_1(t) = + g \cdot dw(t) \\
&dx_2(t) = x_1(t) dt \\
&dy(t) = + x_2(t) dt,
\end{align*} \]

(28)

where

\[ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \]

\[ G = \begin{bmatrix} g \\ 0 \end{bmatrix}, \]

\[ H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]

\( Q = 0 \), hence \( QQ^T \) is surely singular.

Our aim is to estimate the stochastic non-measurable state vector \( x(t) \) from the measurable output \( y(t) \) of the system given by Eq. (28) and Figure 1.
axioms of the new approach to system theory we are able to modify the class of admissible estimators while the Kalman filter is being designed. Firstly, we are to find a sequence of discrete-time systems converging to the given continuous-time system according to the new approach to system theory.

The appropriate sequence of discrete-time systems can be described as

\[ x_k(t) = A_k \cdot x_k(t - h_k) + \Gamma_k \cdot \xi_k(t) \]

\[ y_k(t) = C_k \cdot x_k(t - h_k) + D_k \cdot y_k(t - h_k) , \]

where

\[ A_k = e^{F \cdot h_k} = I + F \cdot h_k = \begin{bmatrix} 1 & 0 \\ h_k & 1 \end{bmatrix} ; \]

\[ C_k = H \cdot h_k = \begin{bmatrix} 0 & h_k \end{bmatrix} ; \]

\[ D_k = 1 ; \]

\[ \Gamma_k = G \cdot \sqrt{h_k} = \begin{bmatrix} h_k \cdot g \\ 0 \end{bmatrix} . \]

This sequence of discrete-time systems formally converges to the given continuous-time system Eq. (28).

\[ F = \frac{1}{h_k} (A_k - I) , \]

\[ \lim_{k \to +\infty} [C_k] = H \cdot dt , \]

\[ G = \frac{1}{\sqrt{h_k}} \Gamma_k , \]

\[ dy(t) = \lim_{k \to +\infty} \left[ y_k(t) - y_k(t - h_k) \right] = \lim_{k \to +\infty} \left[ C_k \cdot x_k(t) \right] = H \cdot x(t) \cdot dt , \]

\[ dx(t) = \lim_{k \to +\infty} \left[ x_k(t) - x_k(t - h_k) \right] = Fx(t)dt + Gdw(t) , \]

where

\[ \lim_{k \to +\infty} \sqrt{h_k} \cdot \xi_k(t) = dw(t) , \]

as described in (Žampa, 2000).

Now, it is possible to design the discrete Kalman filter for the sequence given by Eq. (29). The optimal estimation is given by next equation

\[ \mu_k(t) = A_k \mu_k(t - h_k) + K \left[ y_k(t) - C_k \mu_k(t - h_k) - y_k(t - h_k) \right] . \]

(30)

with the optimal Kalman gain

\[ K_{t+h_k} = (A_k P_k C_k^T) \cdot (C_k P_k C_k^T)^{-1} \]

(31)

and estimation covariance

\[ P_{t+h_k} = A_k P_k A_k^T + \Gamma_k \Gamma_k^T - (A_k P_k C_k^T) \cdot (C_k P_k C_k^T)^{-1} \cdot (C_k P_k C_k^T) . \]

(32)

Since the steady solution of estimation covariance is given as

\[ P_k = \begin{bmatrix} 2g \cdot h_k & g \cdot h_k^2 \\ g \cdot h_k^2 & g \cdot h_k^3 \end{bmatrix} , \]

we are able to compute the optimal Kalman gain
Finally, the optimal estimation \( \mu_k(t) \) of the non-measurable stochastic vector \( x_k(t) \) can be rewritten into components of the vector \( \mu_k(t) \) with correspondent initial conditions \( \mu_k(t = 0) \) as follows

\[
\mu_k^{(1)}(t) = \frac{1}{h_k^2} \left( y_k(t) - 2y_k(t - h_k) + y_k(t - 2h_k) \right),
\]

\[
\mu_k^{(2)}(t) = h_k \cdot \mu_k^{(1)}(t) + \frac{1}{h_k} \left( y_k(t) - y_k(t - h_k) \right).
\]

The discrete estimation task has been just solved. Equations (33) and (34) constitute the discrete-time Kalman filter, which is the solution of the discrete-time estimation problem. Equations (35) and (36) present the optimal estimations of the non-measurable vector components \( x_k^{(1)}(t) \) and \( x_k^{(2)}(t) \). The continuous-time solution is then given by the limits of these equations; it is then described by

\[
\mu(t) = \lim_{k \to +\infty} \mu_k(t) = \left[ \frac{d^2 y(t)}{dt^2}, \frac{dy(t)}{dt}, \frac{dy(t)}{dt} \right],
\]

\[
K = \lim_{k \to +\infty} K_k = \lim_{k \to +\infty} \left[ A_k P_k C_k^T \cdot (C_k P_k C_k^T)^{-1} \right] = \left[ \frac{1}{h_k^2}, \frac{1}{h_k}, \frac{1}{h_k} \right],
\]

\[
P = \lim_{k \to +\infty} P_k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\]

The original continuous-time estimation problem has been thus solved. The class of admissible estimators was extended with left-sided differentiators. The singular estimation problem has been solved with strict abidance of the new approach to system theory basic axioms. Although some engineers are reluctant to differentiators because of their physical feasibility, the presented solution uses the left-sided differentiators of an appropriate order and therefore there are no problems with their feasibility any more. Considering the integrative character of the given continuous-time system the left-sided derivatives are equal to the right-sided ones and thus to derivatives.

The existence conditions of differentiator possible utilizing in cybernetics were set in (Žampa, Arnošt, 1999) where other advantages of the recently submitted new approach to system theory were shown as well.

5 Conclusions

The process of continualisation was presented in this paper in terms of new approach to system theory (Žampa, 1999). The importance of the new view at cybernetic systems was emphasized by the presented application, where the singular estimation problem was solved thanks to the strict abidance of basic axioms of the new approach to system theory.

All system variables have to be defined, from principal reasons, on finite sets only. Moreover, under carefully chosen continuity hypothesis an extension to infinite sets is straightforward.
and brings new important results featuring adequate description of real systems. However, the continualisation procedure must provide the limit in the mean (Srinivasan, Mehata, 1988) in case of the stochastic cybernetic systems instead of the pure limit in case of the deterministic (theoretic) cybernetic systems.

Even though some engineers are reluctant to continuous-time differentiators because of their impossible physical feasibility, left-sided differentiators were used to get an optimal estimation of non-measurable inner system variables in this paper according to the principle of causality law. Therefore, there are no problems with their feasibility any more.

The new approach to system theory provides modern solution of various cybernetic tasks, it seems to be sufficiently correct and with a good agreement with observation of real world. Important progress was carried out in a study of system structure. A new insight into system properties was gained and many system theory puzzles clarified. The theory can be also useful as a methodology of model building and simulation and may contribute to system terminology unification.

9 References
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